4. VAN DYKE M., Perturbation Methods in Fluid Mechanics. Academic Press, New York, 1964.
5. BATEMAN H. and ERDELYI A., Higher Transcendental Functions, McGraw-Hill, New York, 1955.

Translated by L.K.

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A GENERALIZED FAXEN FORMULA FOR VARIOUS FORMS OF BOUNDARY CONDITIONS*

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The method of integrating the boundary conditions in the Stokes approximation is used to obtain expressions for the forces of resistance of a spherical drop with the usual boundary conditions taking both the surface viscosity and changes in surface tension into account, as well as that of a solid sphere with boundary conditions of slippage.

Faxen established formulas for the force of resistance and momentum acting on a solid sphere with boundary conditions of adhesion, for the case when the sphere moves and rotates in an arbitrary Stokes flow /1/ (satisfying Stokes's equations). The result was generalized in /2/ to the case of a spherical drop, using the Hadamard-Rybchinskii equation and the reciprocity theorem for the Stokes flows generalized in /3/.

Below, a relatively simple method is presented for determining the forces acting on a spherical particle in an inhomogeneous Stokes flow. The perturbation fields introduced into the flow by the particle are described by a Lamb series /4/. Subsequent integration over the surface of the sphere of the boundary conditions specified on its surface enables us to determine the required integral characteristics in terms of which the force acting on the particle is expressed. The final formulas contain the integrals of the characteristics of the inhomogeneous flow impinging on the sphere, and represent a generalization of the Faxen formulas /1/.

1. When an arbitrary Stokes flow moves past a sphere, a perturbation field described by a Lamb series appears by virtue of the need to satisfy the boundary conditions on the sphere. The force of resistance acting on the sphere is found to depend only on the stresses caused by the presence of the perturbation field. It can be shown that integration of the stresses present in the basic flow over the whole surface of the sphere gives a zero result for any Stokes flow. The contribution of the perturbation field will depend only on the function p_{-2} (the harmonic function appearing in Lamb's solution /4/). The remaining terms of the Lamb expansion make no contributon to the integral expressing the force of resistance D, by virtue of the orthogonality of spherical functions of various orders on the sphere. Hence we obtain

$$\mathbf{D} = -3a^{-1}\int_{\Sigma} p_{-2}\mathbf{r}\, ds \tag{1.1}$$

where a is the radius, Σ is the surface of the sphere, and $\mathbf{r}$ is the radius vector drawn from the centre of the sphere to a point on its surface.

Let the sphere be situated in an incoming inhomogeneous flow $\mathbf{v}^\infty$ determined relative to a frame of reference attached to the sphere. Then, using the boundary conditions on the sphere, we can be obtain a system of equations from which the value of the integral (1.1) can be found.

Let us consider a liquid sphere on whose surface the following conditions must hold: the total radial velocities of the external flow (index e) and internal flow (index i) $u_r^e = 0$; $u_r^i = 0$ must vanish; the tangential stresses $\mathrm{P}_{r\tau}^e = \mathrm{P}_{r\tau}^i$ are continuous; the total tangential velocities $u_\tau^e = u_\tau^i$ are continuous. The velocity perturbations due to the presence of a spherical drop streamlined by an inhomogeneous flow $\mathbf{v}^\infty$ are described by the Lamb series in

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harmonic functions of negative order (p_{-m}, φ_{-m}) for perturbations of the external flow $\mathbf{v}^e$ and of positive order (p_n, φ_n) for perturbations of the internal flow $\mathbf{v}^i$. Thus the total velocities are expressed by the equations $\mathbf{u}^e = \mathbf{v}^\infty + \mathbf{v}^e$, $\mathbf{u}^i = \mathbf{v}^i$. Integrating the boundary conditions formulated over the surface of the sphere, we obtain the following system of four linear relations connecting the four vector integrals:

$$\mathbf{X} = \int p_{-2}\mathbf{r}\, ds, \quad \mathbf{Y} = \int \varphi_{-2}\mathbf{r}\, ds \tag{1.2}$$
$$\mathbf{Z} = \int p_1\mathbf{r}\, ds, \quad \mathbf{W} = \int \varphi_1\mathbf{r}\, ds$$

Here and henceforth all integrals will be taken over the surface of the sphere. Solving this system we obtain

$$\mathbf{X} = -\frac{\sigma\mu^e}{2(1+\sigma)}\int \mathbf{v}^\infty ds - \frac{\mu^e}{1+\sigma}\int \mathbf{v}_r{}^\infty ds + \frac{a}{6(1+\sigma)}\int \mathbf{P}_{r\tau}^\infty ds \tag{1.3}$$

where μ^e and μ^i are the viscosities of external and internal fluid respectively, $\sigma = \mu^i/\mu^e$, $\mathbf{P}_r{}^\infty$ is the stress vector in the incoming flow on a surface with normal r, and $\mathbf{P}_{r\tau}{}^\infty = \mathbf{P}_r{}^\infty - \mathbf{P}_{rr}{}^\infty$, $\mathbf{P}_{rr}{}^\infty = (\mathbf{P}_r{}^\infty \cdot \mathbf{r}r^{-1})\mathbf{r}r^{-1}$.

Further, according to formula (1.1) the resistance in the case of an incoming Stokes flow has the form

$$\mathbf{D} = \frac{3\mu^e\sigma}{2a(1+\sigma)}\int \mathbf{v}^\infty ds + \frac{3\mu^e}{a(1+\sigma)}\int \mathbf{v}_r{}^\infty ds - \frac{1}{2(1+\sigma)}\int \mathbf{P}_{r\tau}^\infty ds \tag{1.4}$$

In the case of a solid particle we have $\sigma \to \infty$, and for a bubble we have $\sigma \to 0$ * (*The formula for $\sigma \to 0$ is identical with the expression obtained in the paper by Struminskii V.V. Smirnov L.P., Kul'bitskii YU.I., Gus'kov O.B. and Korol'kov G.A., Laws of mechanics of disperse media and two-phase systems in connection with the problems of increasing the efficiency of technological processes. The method of classical mechanics. Preprint 1, Moscow, Section on the Mechanics of Inhomogeneous Media of the Academy of Sciences of the USSR, 1979.)

$$\mathbf{D} = \frac{3\mu^e}{2a}\int \mathbf{v}^\infty ds \quad (\sigma \to \infty)$$
$$\mathbf{D} = \frac{3\mu^e}{a}\int \mathbf{v}_r{}^\infty ds - \frac{1}{2}\int \mathbf{P}_{r\tau}^\infty ds \quad (\sigma \to 0)$$

Comparing formula (1.4) with a relation from /2/, we obtain

$$\mathbf{D} = 2\pi\mu^e a\frac{2+3\sigma}{1+\sigma}(\mathbf{v}^\infty)_0 + \pi a^3\frac{\sigma}{1+\sigma}(\nabla p)_0$$

(where the zero subscript indicates the value of the quantity in question at the origin of coordinates, i.e. at the centre of the sphere). We can show that they are identical, provided that the integral expressions in (1.4) are expanded in a Taylor series about the centre of the sphere, using the technique described in /5/.

2. We shall use two more examples to show what results can be obtained using the proposed method with different boundary conditions, so as to demonstrate its universality.

Let us consider a solid medium in an inhomogeneous flow with boundary conditions of slippage. Using a frame of reference rigidly coupled to the sphere, we obtain the following boundary conditions at its surface: $\mathbf{v}_r{}^\infty + \mathbf{v}_r = 0$, $\mathbf{v}_\tau{}^\infty + \mathbf{v}_\tau = (\varkappa/\mu^e)(\mathbf{P}_{r\tau}{}^\infty + \mathbf{P}_{r\tau})$, where χ is the coefficient of slippage.

Integrating the boundary conditions over the surface of the sphere, we obtain a system of equations connecting the integrals $\mathbf{X}$, $\mathbf{Y}$ (1.2), and solving this system we obtain with help of formula (1.1),

$$\mathbf{D} = -\frac{3\varkappa}{2a(1+3\varkappa/a)}\int \mathbf{P}_{r\tau}^\infty ds + \frac{3\mu^e}{2(a+3\varkappa)}\int \mathbf{v}^\infty ds + 9\frac{\mu^e\varkappa/a}{a(1+3\varkappa/a)}\int \mathbf{v}_r{}^\infty ds \tag{2.1}$$

The corresponding limit values are obtained from formula (2.1) as $\varkappa \to 0$ (adhesion) and as $\varkappa \to \infty$ (the boundary conditions on the surface of the bubble).

3. We shall now consider the boundary conditions arising in the case when the surface tension, which varies over the surface of the drop, must be taken into account. The first three conditions on the surface of the drop remain unchanged: $v_r{}^\infty + v_r{}^e = 0$, $v_r{}^i = 0$, $\mathbf{v}_\tau{}^\infty + \mathbf{v}_\tau{}^e = v_\tau{}^i = v^\circ$.

The fourth condition now includes the gradient of the surface tension γ (axial symmetry of the incoming flow and an axisymmetric distribution of the surface tension are both assumed) and the surface viscosity ξ /6/

$$P_{r\tau}^{\infty} + P_{r\tau}^{e} - P_{r\tau}^{i} = -\frac{1}{a}\frac{d\gamma}{d\theta} - \frac{\xi}{a^2}\frac{\partial^2 v^\circ}{\partial\theta^2}$$

We will add a fifth condition expressing the equilibrium of the normal forces applied to the surface element of the sphere, including the normal component governed by the surface tension

$$P_{rr}^{\infty} + P_{rr}^{e} - P_{rr}^{i} = 2\gamma/a$$

We shall assume that in the flow past a spherical drop a gravitational force field acts on the external and internal fluid. The field balances the resultant hydrodynamic resistance. The Stokes equations for the external and internal flow can be written using a frame of reference in which the z axis coincides with the direction of gravity, in the form

$$\mu^e \Delta \mathbf{v}^e = \nabla \Pi^e, \quad \mu^i \Delta \mathbf{v}^i = \nabla \Pi^i \quad (\Pi^k = p^k - \rho^k g z, \quad k = e, i) \tag{3.1}$$

where g is the acceleration due to gravity. Integrating the boundary conditions over the surface of the sphere, we obtain

$$\int \mathbf{v}_r^{\infty}\, ds + \frac{1}{\mu^e}\mathbf{X} - \frac{2}{a^2}\mathbf{Y} = 0$$

$$\frac{1}{10\mu^i}\mathbf{Z} + \frac{1}{a^2}\mathbf{W} = 0$$

$$\int \mathbf{v}_\tau^{\infty}\, ds + \frac{1}{\mu^e}\mathbf{X} + \frac{2}{a^2}\mathbf{Y} = \frac{2}{5\mu^i}\mathbf{Z} + \frac{2}{a^2}\mathbf{W}$$

$$\int \mathbf{P}_{r\tau}^{\infty}\, ds - 12\frac{\mu^e}{a^3}\mathbf{Y} = \frac{3}{5a}\mathbf{Z} - \frac{1}{a}\int \frac{d\gamma}{d\theta}\boldsymbol{\tau}_0\, ds - \frac{2\xi}{5\mu^i a^2}\mathbf{Z} - \frac{2\xi}{a^4}\mathbf{W}$$

$$\int \mathbf{P}_{rr}^{\infty}\, ds - \frac{3}{a}\mathbf{X} + 12\frac{\mu^e}{a^3}\mathbf{Y} + \frac{3}{5a}\mathbf{Z} + \frac{4}{3}\pi a^3 g(\rho^i - \rho^e)\mathbf{k} = \frac{2}{a^2}\int \gamma \mathbf{r}\, ds$$

where $\boldsymbol{\tau}_0$ is the unit vector of the tangent to the meridional cross-section of the sphere in the direction of increasing angle θ (θ is measured from the leading stagnation point), and $\mathbf{k}$ is the unit vector of the z axis.

After eliminating from the system of equations the four unknown vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}$ we obtain the following integral relation which must be satisfied by the surface tension of the drop retaining its spherical form:

$$\int \mathbf{P}_{rr}^{\infty}\, ds + \frac{9\mu^e + 6\xi/a}{a\beta}\int \mathbf{v}_r^{\infty}\, ds + \frac{9\mu^i + 6\sigma\xi/a}{2a\beta}\int \mathbf{v}^{\infty}\, ds + \frac{3(2\sigma+1)}{2\beta}\times \tag{3.2}$$
$$\left\{\int \mathbf{P}_{r\tau}^{\infty}\, ds + \frac{1}{a}\int \frac{d\gamma}{d\theta}\boldsymbol{\tau}_0\, ds\right\} + \frac{4}{3}\pi a^3 g(\rho^i - \rho^e)\mathbf{k} = \frac{2}{a^2}\int \gamma\mathbf{r}\, ds; \quad \beta = 3 + 3\sigma + \xi/\mu^e a$$

For the vector $\mathbf{X}$ we have

$$X = \frac{a}{2\beta}\left\{\int \mathbf{P}_{r\tau}^{\infty}\, ds + \frac{1}{a}\int \frac{d\gamma}{d\theta}\boldsymbol{\tau}_0\, ds\right\} - \frac{3\mu^e}{\beta}\int \mathbf{v}_r^{\infty}\, ds - \frac{3\mu^i}{2\beta}\int \mathbf{v}^{\infty}\, ds \tag{3.3}$$

and for the homogeneous flow $\mathbf{v}^{\infty} = -U\mathbf{k}$ relation (3.3) takes the form

$$\mathbf{D} = 6\pi\mu^e a\mathbf{U}\,\frac{3\sigma + 2 + \xi_*}{3\sigma + 3 + \xi_*} - \frac{3}{2a(3\sigma + 3 + \xi_*)}\int \frac{d\gamma}{d\theta}\boldsymbol{\tau}_0\, ds, \quad \xi_* = \frac{\xi}{a\mu^e} \tag{3.4}$$

When the surface tension is constant, formula (3.4) is identical with the expression for the resistance force obtained by Boussinesq /6/. The expression for the resistance force obtained in /7/ is a special case of formula (3.4) when $U = 0$ and $\xi = 0$.

In the case of a cosine dependence of the surface tension γ on the angle θ without taking into account the surface viscosity $(\gamma = \gamma_0 + \alpha\cos\theta, \xi = 0)$, relation (3.2) can be written, when $\mathbf{v}^{\infty} = \mathbf{U} = -U\mathbf{k}$, in the form

$$-\frac{1+2\sigma}{2a(1+\sigma)}\int \alpha\sin\theta\boldsymbol{\tau}_0\, ds + \frac{3\mu^i}{2a(1+\sigma)}4\pi a^2\mathbf{U} + \frac{3\mu^e}{a(1+\sigma)}\frac{4}{3}\pi a^2\mathbf{U} + \tag{3.5}$$
$$\frac{4}{3}\pi a^3 g(\rho^i - \rho^e)\mathbf{k} = \frac{2}{a^2}\int \alpha\cos\theta\mathbf{r}\, ds$$

Using the relations

$$\int \sin\theta\tau_\theta\, ds = -\frac{8\pi a^2}{3}\mathbf{k}, \quad \int \cos\theta\mathbf{r}\, ds = \frac{4}{3}\pi a^3\mathbf{k}$$

and the fact that the vectors $\mathbf{U}$ and $\mathbf{k}$ are in opposite directions, we obtain

$$\frac{4}{3}\pi a^3 g(\rho^i - \rho^e) = 6\pi\mu^e aU\frac{\sigma + {}^2/_3}{\sigma+1} + \frac{4\pi a\alpha}{3(1+\sigma)} \tag{3.6}$$

This result shows that the difference between the weight ad the Archimedean force balances the reaction found from the Hadamard-Rybchinskii formula, summed with an additional term proportional to the coefficient α. At the same time, we find that from relation (3.3) with $\xi = 0$ and condition of uniformity of the flow,

$$\mathbf{X} = \frac{4\pi a^2}{9(1+\sigma)}\alpha\mathbf{k} - \left(\frac{\mu^i}{2} + \frac{\mu^e}{3}\right)\frac{4\pi a^2}{1+\sigma}\mathbf{U}$$

We further have $\mathbf{D} = -3a^{-1}\mathbf{X}$, and we obtain for D a value equal to the right-hand side of relation (3.6). Thus the result (3.6) obtained from the integral relation (3.5) agrees with the result of direct calculation of the resistance force.

Let us substitute expression (3.3) into (3.2), remembering the relation $\mathbf{D} = -3a^{-1}\mathbf{X} = -{}^4/_3\pi a^3 g(\rho^i - \rho^e)\,\mathbf{k}$, and put $\xi = 0$. This yields a condition imposed on the distribution of the surface tension for a spherical drop

$$\int \mathbf{P}_r^\infty\, ds + \frac{1}{a}\int \frac{d\gamma}{d\theta}\tau_\theta\, ds = \frac{2}{a^2}\int \gamma\mathbf{r}\, ds \tag{3.7}$$

In the case of an incoming Stokes flow, and in particular when the flow is uniform, the first term on the right-hand side of (3.7) vanishes. When the distribution of the surface tension is axisymmetric, this yields

$$\int\left(\frac{d\gamma}{d\theta}\sin\theta + 2\gamma\cos\theta\right)ds = 0 \tag{3.8}$$

It can be shown that any function $\gamma(\theta)$, which can be expanded in a trigonometric series in the sines and cosines of multiple arcs, will satisfy relation (3.8).

We have used, as an example, the generalizations of the Faxen formula (1.4) and (2.1) to determine the resistance forces of two spheres, taking into account their hydrodynamic interaction /8/.

The expressions in /8/ for the resistance forces of two liquid spheres contain errors. Below we give the correct version of formula (7) or /8/ for the resistance forces of two liquid spheres of equal radius

$$D_a = D_b = 6\pi\mu^e aU\lambda_1\left(1 + \frac{1}{2}\lambda_2\frac{a^3}{d^3}\right)\left(1 + \frac{3}{2}\lambda_1\frac{a}{d} - \frac{1}{2}\lambda_2\frac{a^3}{d^3}\right)^{-1}$$

$$\lambda_1 = \frac{\sigma + \frac{2}{3}}{\sigma+1}, \quad \lambda_2 = \frac{\sigma}{\sigma+1}$$

If the radii are different, then the resistances of the spheres D_a, D_b are given by a system of two equations, one of which has the form

$$D_a = 6\pi\mu^e aU\lambda_1 + 3\pi\mu^e aU\lambda_1\lambda_2\frac{b^3}{d^3} - \left(\frac{3}{2}\frac{a}{d} - \frac{1}{2}\frac{a^3}{d^3}\right)\lambda_1 D_b - \frac{1}{1+\sigma}\frac{a}{d}D_b$$

while the second is obtained by replacing a by b and b by a.

REFERENCES

1. FAXEN H., Der Widerstand gegen die Bewegung einer starren Kugel in einer zähen Flüssigkeit, die zwischen zwei parallelen, ebenen Wänden eingeschlossen ist, Ark. Mat. Astron. Fis. 18, 29, 1924.
2. HETSRONI G., WACHOLDER E. and HABER S., The hydrodynamic resistance of a fluid sphere submerged in Stokes flows. ZAMM, 51, 1, 1971.

3. HAPPEL J. and BRENNER H., Low Reynolds Number Hydrodynamics. Noordhoff Intern. Publ. Leiden, 1973.
4. LAMB H., Hydrodynamics. Dover, New York, 1945.
5. BRENNER H., The Stokes resistance of an arbitrary particle. IV Arbitrary fields of flow. Chem. Eng. Sci. 19, 10, 1964.
6. BOUSSINESQ I., Sur l'existence d'une viscosité superficielle dans la mince couche de transition separant un liquide d'un autre fluide contignue. Ann. Chim. Phys., 29, 1913.
7. SUBRAMANIAN R.S., The Stokes force on a droplet in an unbounded fluid medium due to capillary effects. J. Fluid Mech. 153, 1985.
8. AVAKYAN V.A. and SMIRNOV L.P., Resistance of two liquid, solid and gaseous spheres in a homogeneous axisymmetric flow, taking their interaction into acccount. Vestn. MGU, Ser.1, Matematika, Mekhanika, 1, 1987.

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THE CONSTRUCTION OF THE CONSTANT-VELOCITY CONTOUR OF A FOUNDATION OF A HYDRAULIC INSTALLATION IN THE CASE OF THE FILTRATION OF TWO LIQUIDS OF DIFFERENT DENSITY*

E.N. BERESLAVSKII

An underground, constant-velocity contour is constructed for the case when a layer of stagnant salt water forms at a certain depth in a flow of water under a dyke. Results of numerical computations are presented and an analysis given of the influence of the fundamental defining parameters of a model on the form and size of the underground contour of a dam. Limiting cases of flows are mentioned, namely the scheme with a water-confining stratum /1/ and filtration around a point channel /2-4/.

1. Formulation of the problem. Consider the steady plane flow of fresh water of density ρ_1 under an underground impermeable contour of a channel BC in the case when a layer of salt water of density $\rho_2\,(\rho_2>\rho_1)$ appears at a certain depth above an impermeable layer of salt. The domain of filtration z (Fig.1) is bounded from below by a boundary AD passing through a fixed point $z_0=-ih_0$ where h_0 is the depth of the initial surface (before the squeeze) of salt water. The pressure H acting on the installation and the width of the flood bed l, whose left-hand end is fixed at the point $B\,(z=-l_1)$ are assumed given, and the boundaries of the head and tail by AB and CD are horizontal. The flow obeys D'Arcy's law, and the soil is assumed to be homogeneous and isotropic.

Let us introduce the complex potential $\omega=\varphi+i\psi$ and complex coordinate $z=x+iy$ referred, respectively, to $\varkappa h_0$ and h_0, where $\varkappa$ is the soil filtration coefficient. Let us put $\varphi=-H/2$ on AB, $\varphi=H/2$ on CD and $\omega=Q$ along the water-impermeable contour of the flood bed BC, where Q is the filtration flow rate. Then we find that the following conditions must hold at the boundary line AD:

$$\varphi-cy=\mathrm{const},\ \psi=0\ (c=\rho_2/\rho_1-1) \tag{1.1}$$

The first relation of (1.1) for the segment AD follows from the assumption that salt water is stagnant and the pressure remains continuous during the passage across the boundary line /5, 6/. The condition of continuity of the potential at infinity to the left and right, together with condition /5, 6/ $h_0=(h_1+h_2)/2$, which follows from the assumptions concerning the incompressibility of the liquid, determine the value of the constant in condition (1.1), and the difference in depth to the left and right after squeezing $h_1-h_2=H/c$. From this it follows that

$$h_1=h_0+H/(2c),\ h_2=h_0-H/(2c) \tag{1.2}$$

and this determines the region of flow of the ground water.

Next it is required to construct an underground contour BC, so that the filtration rate

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